THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 10
17th November2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Determine whether the following functions are uniformly continuous. Justify your answers.
   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ for $n \geq 2$.
   (b) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{1/n}$.
2. Let $f : (a, b) \to \mathbb{R}$ be a uniformly continuous function, show that it is bounded.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, if f^2 is uniformly continuous, is it true that f is also uniformly continuous?
4. Suppose $f, g : A \to \mathbb{R}$ are bounded, uniformly continuous functions, prove that fg is also uniformly continuous. Provide a counter-example to the claim if the boundedness assumption is dropped.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, i.e. there exists some $p > 0$ so that $f(x + p) = f(x)$ for any $x \in \mathbb{R}$. Prove that f is bounded and uniformly continuous.
6. Let $f : A \to \mathbb{R}$ be a function, for any $\delta > 0$, define
   $$\omega_f(\delta) := \sup\{|f(x) - f(y)| : x, y \in A, |x - y| < \delta\}.$$
   Show ω is called the modulus of continuity of f.
   (a) Prove that f is uniformly continuous on A if and only if $\lim_{\delta\to 0} \omega_f(\delta) = 0$.
   (b) Compute $\omega_f(\delta)$ for the function $f(x) = \sin(1/x)$ defined on $(0, 1)$. What does it say about the uniform continuity of f?
   (c) If f is a uniformly continuous function on $A = \mathbb{R}$, prove that $\omega_f(\delta)$ when regarded as a function $\omega_f : [0, \infty) \to [0, \infty)$, is uniformly continuous. (Hint: Try to show $\omega_{\omega_f} \leq \omega_f$.)
7. Let $f : [-1, 0) \cup (0, 1] \to \mathbb{R}$ be a continuous function, prove that it is uniformly continuous if and only if $\lim_{x\to 0} f$ exists.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that $\lim_{n\in\mathbb{N}, n\to\infty} f(n) = 0$. First come up with an example where $\lim_{x\to\infty} f(x) \neq 0$. Now further assume that $f(x^2)$ is uniformly continuous, prove that $\lim_{x\to\infty} f(x) = 0$.